Free Vibrations of TDOF Systems: Analysis

The oscillating beams illustrated in Figure Q1 each have a moment of inertia of J about their pivots. Derive the differential equations of motion and write them in matrix notation. Further, solve for the natural mode frequencies and the corresponding amplitude ratios.

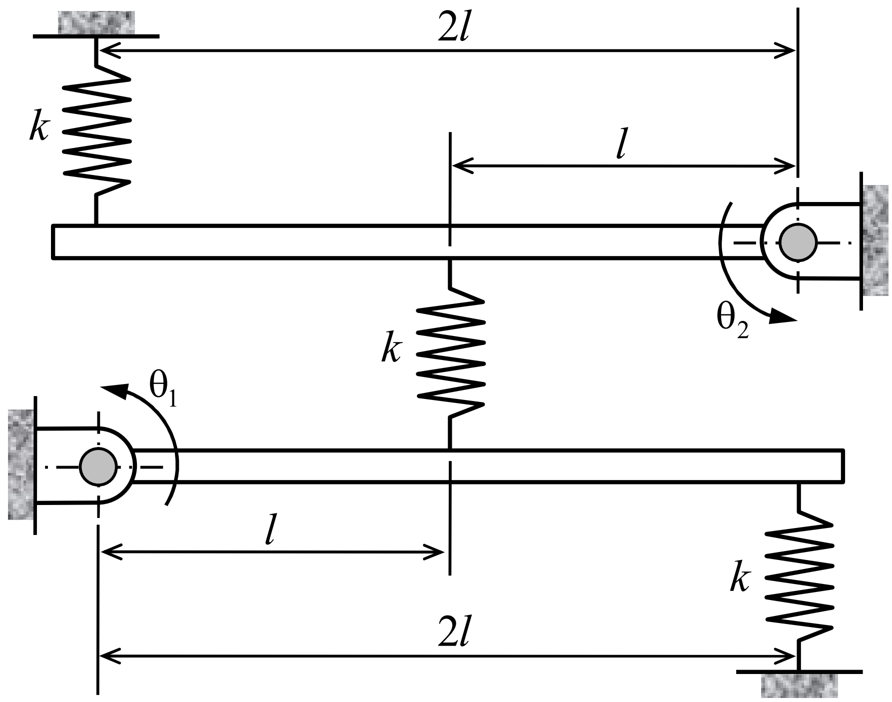


Figure Q1

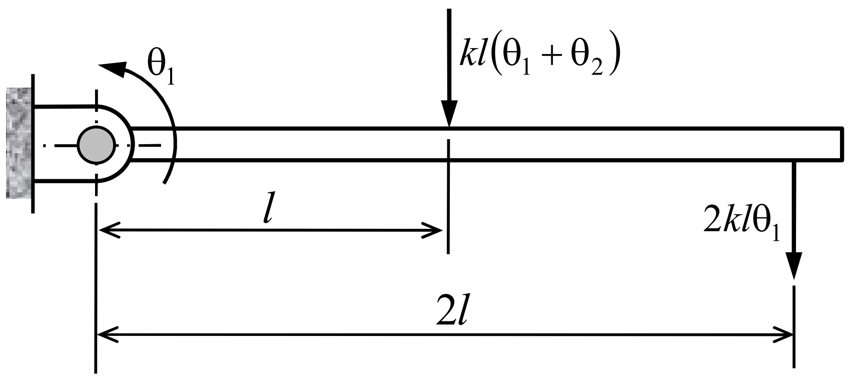
Solution

The free body diagram of the lower beam is illustrated below. Applying Newton's second law of motion to the lower beam leads to the following equation:

$$\left.\begin{aligned} J\ddot{\theta}_1 &= -kl^2(\theta_1+\theta_2) - 4kl^2\theta_1 \\ &= -5kl^2\theta_1 - kl^2\theta_2 \end{aligned}\right\}$$

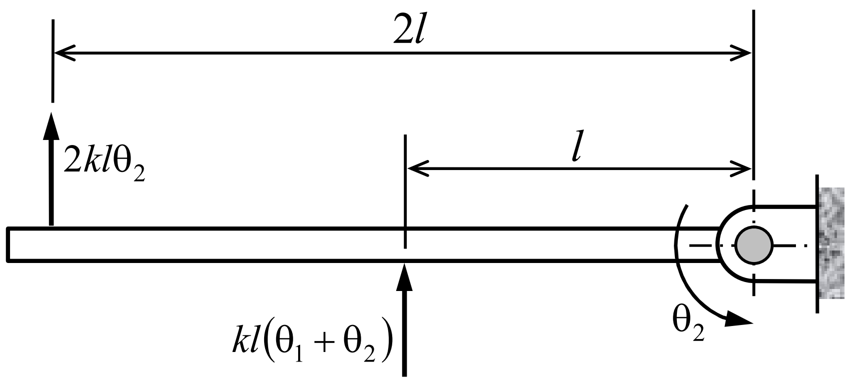
Thus, the equation of motion for the lower beam is as follows:

$$J\ddot{\theta}_1 + 5kl^2\theta_1 + kl^2\theta_2 = 0 \tag{1}$$



Free Body Diagram of the Lower Beam

The free body diagram of the upper beam is also shown below.



Free Body Diagram of the Upper Beam

Applying Newton's second law of motion to the upper beam leads to the following equation:

$$\left.\begin{aligned} J\ddot{\theta}_2 &= -kl^2(\theta_1+\theta_2) - 4kl^2\theta_2 \\ &= -kl^2\theta_1 - 5kl^2\theta_2 \end{aligned}\right\}$$

Thus, the equation of motion for the lower beam is as follows:

$$J\ddot{\theta}_2 + kl^2\theta_1 + 5kl^2\theta_2 = 0 \tag{2}$$

The equations of motion, written in matrix notation are the following:

$$\begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} 5kl^2 & kl^2 \\ kl^2 & 5kl^2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3}$$

In the natural modes, the solutions to the equations of motion may be represented as follows:

$$\left.\begin{aligned}\theta_i(t) &= A_i \sin\omega_n t \\ \ddot{\theta}_i(t) &= -\omega_n{}^2 A_i \sin\omega_n t\end{aligned}\right\} \tag{4}$$

Substituting equations (4) into equations (1) and (2) leads to the following:

$$\left.\begin{aligned}\left(5kl^2 - \omega_n{}^2 J\right)A_1 + kl^2 A_2 = 0 \\ kl^2 A_1 + \left(5kl^2 - \omega_n{}^2 J\right)A_2 = 0\end{aligned}\right\} \tag{5}$$

Equations (5) may be re-written as a single matrix equation, as follows:

$$\begin{bmatrix}\left(5kl^2 - \omega_n{}^2 J\right) & kl^2 \\ kl^2 & \left(5kl^2 - \omega_n{}^2 J\right)\end{bmatrix}\begin{Bmatrix}A_1 \\ A_2\end{Bmatrix} = \begin{Bmatrix}0 \\ 0\end{Bmatrix} \tag{6}$$

Application of Cramer's rule to the above equation leads to the following:

$$\left.\begin{aligned}\left(5kl^2 - \omega_n{}^2 J\right)^2 - k^2 l^4 = 0 \\ J^2\omega_n{}^4 - 10Jkl^2\omega_n{}^2 + 24k^2 l^4 = 0\end{aligned}\right\} \tag{7}$$

Equation (7) can now be solved for the two natural frequencies, as follows:

$$\left.\begin{aligned}\omega_{n1}{}^2 &= \frac{10Jkl^2 + \sqrt{100J^2k^2l^4 - 96J^2k^2l^4}}{2J^2} \\ &= \frac{10Jkl^2 + 2Jkl^2}{2J^2} = \frac{6kl^2}{J} \\ \omega_{n1} &= \sqrt{\frac{6kl^2}{J}}\end{aligned}\right\} \tag{8}$$

Similarly:

$$\left.\begin{aligned}\omega_{n2}{}^2 &= \frac{10Jkl^2 - 2Jkl^2}{2J^2} = \frac{4kl^2}{J} \\ \omega_{n2} &= \sqrt{\frac{4kl^2}{J}}\end{aligned}\right\} \tag{9}$$

To obtain the corresponding ratios of amplitudes, the expressions of natural mode frequencies can be substituted into either of the equations (5). Thus:

$$\left.\begin{aligned}\left(5kl^2 - 6kl^2\right)A_{11} &= -kl^2 A_{21} \\ u_1 = \frac{A_{11}}{A_{21}} = \frac{-kl^2}{-kl^2} &= 1\end{aligned}\right\} \tag{10}$$

Similarly:

$$\left.\begin{aligned}\left(5kl^2 - 4kl^2\right)A_{12} &= -kl^2 A_{22} \\ u_2 = \frac{A_{12}}{A_{22}} = \frac{-kl^2}{kl^2} &= -1\end{aligned}\right\} \tag{11}$$